Two Two-dimensional Terminations

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1 Introduction

Varieties with log terminal and log canonical singularities are considered in the Minimal Model Program, see [KMM] for introduction. In [SH2] it was conjectured that many of the interesting sets, associated with these varieties have something in common: they satisfy the ascending chain condition, which means that every increasing chain of elements terminates (in [SH2] it was called the upper semi-discontinuaty). Philosophically, this is the reason why two main hypotheses in the Minimal Model Program: existence and termination of flips should be true and are possible to prove.

As for the latter, one of the main properties of flips is that log discrepancies after doing one do not decrease and some of them actually increase, [SH1]. Therefore, if one could show that a set of "the minimal discrepancies" satisfies the ascending chain condition, that would help to prove the termination of flips. The Shokurov's proof of existence of 3-fold log flips [SH3] is another example of applying the same principle. In fact, to complete the induction it uses some 1 - dimensional statement, 2 - dimensional analog of which is proved in this paper. For further discussion, see also [A-K].

For one of the first examples where the phenomenon is actually proved let us mention the following

Theorem 1.1 ([A1],[A2]) *Let us define the Gorenstein index of an n-dimensional Fano variety X with weak log terminal singularities as the maximal rational number r such that the anticanonical divisor $-K_X \equiv rH$ with an ample Cartier divisor H. Then a set*

$$FS_n \cap [n-2, = +\infty] = \{r(X) | X \textit{ is a Fano variety and } r(X) > n-2\}$$

satisfies the ascending chain condition and has only the following limit points: $n-2$ *and* $n-2+\frac{1}{k}$, $k=1,2,3....$

In this paper we prove that the following two sets satisfy the ascending chain condition:

(i) (Theorems 3.2,3.8) The set of minimal log discrepancies for K_X+B where X is a surface with log canonical singularities.

(ii) (Theorem 5.3) The set of groups $(b_1,...b_s)$ such that there is a surface X with log canonical and numerically trivial $K_X+\sum b_jB_j$. The order on such groups is defined in a natural way, see 2.26.

The proofs heavily use explicit formulae for log discrepancies from [A3]. We do not find it possible to prove them here again. (This is quite easy anyway).

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2 Definitions and recalling

All varieties in this paper are defined over the algebraically closed field of characteristic zero. K_X or simply K if the variety X is clear from the context always denote the class of the canonical divisor.

2.1 Basics

Definition 2.1 *A* **Q-divisor** *on a variety* X *is a formal combination* $D=\sum d_jD_j$ *of Weil divisors with rational coefficients.*

Definition 2.2 *One says that a* $\mathbf{Q}$*-divisor* D *is* **Q-Cartier** *if some multiple of it is a Weil divisor with integer coefficients that is a Cartier divisor.*

Definition 2.3 *Let $f : Y \to X$ be any birational morphism and F_i be exceptional divisors of this morphism. Consider a divisor of the form $K + B$, where $B = \sum b_j B_j$ and $0 < b_j \leq 1$. Coefficients a_i in the following formula*

$$K_Y + f^{-1}B + \sum F_i = f^*(K + B) + \sum a_i F_i$$

are called **log discrepancies** *of $K + B$.*

Definition 2.4 *Let $f : Y \to X$ be any birational morphism and F_i be exceptional divisors of this morphism. Consider a divisor of the form $K + B$, where $B = \sum b_j B_j$ and $0 < b_j \leq 1$. Coefficients b_i in the following formula*

$$K_Y + f^{-1}B + \sum b_i F_i = f^*(K + B)$$

are called **codiscrepancies** *of $K + B$.*

Remark 2.5 *Evidently there is a simple relation between log discrepancy and codiscrepancy: $a_i = 1 - b_i$.*

Definition 2.6 *A $\mathbf{Q}$-divisor of the form $K + B$ is said to be* **log canonical (lc)** *if*

(i) $K + B$ is $\mathbf{Q}$-Cartier

(ii) there is a resolution of singularities $f : Y \to X$ such that supp$(f^{-1}B) \bigcup F_i$ is a divisor with normal intersections and all the log discrepancies $a_i \geq 0$.

Definition 2.7 *A $\mathbf{Q}$-divisor of the form $K + B$ is said to be* **log terminal (lt)** *if*

(i) $K + B$ is $\mathbf{Q}$-Cartier

(ii) there is a resolution of singularities $f : Y \to X$ such that supp$(f^{-1}B) \bigcup F_i$ is a divisor with normal intersections and all the log discrepancies $a_i > 0$.

2.2 Graphs

With rare exceptions all the varieties in this paper will be two-dimensional. No doubt that the case of surfaces is much easier than that of more-dimensional varieties. One of the reasons for this is that surface has a natural quadratic form defined by intersection of curves. Many statements that we need can be formulated in terms of weighted graphs and become therefore basicly combinatorical problems.

So let us start with a system of curves on a surface that are divided into two classes: "internal", denoted by F_i and "external", denoted by B_j.

Definition 2.8 *A weighted graph* Γ *is the following data:*

(i) a "ground graph": each vertex v *of it corresponds to an "internal" curve* F *, two different vertices* v_1 *and* v_2 *are connected by wedge of weight* $F_1 \cdot F_2$*.*

(ii) weights: a vertex v *has weight* $w = -F^2$

(iii) genera: a vertex v *has genus* $p_a(F)$ *(arithmetical genus of the curve)*

(iv) an "external part": additional vertices, corresponding to the "external" components B_j *, connected with vertices* v_i *if* B_j *and* F_i *intersect.*

Vice versa, every weighted graph Γ corresponds to a system of curves $\{F_i, B_j\}$.

Definition 2.9 *Graph* Γ *is said to be* **elliptic, parabolic or hyperbolic** *if the corresponding quadratic form* $F_i \cdot F_k$ *is elliptic, parabolic or hyperbolic, that is, has the signature* $(0, n)$*,* $(0, n-1)$ *or* $(1, n-1)$*.*

The following is the basic case when we shall need such graphs: X is a surface with a divisor $K + B$ and $f : Y \to X$ is a resolution of singularities of X. The curves F_i are exceptional divisors of f and the curves B_j are strict transforms of the components of B. Note that since a matrix of intersection $(F_i \cdot F_k)$ is negatively defined, the graph is elliptic and all the weights in this case are positive integer numbers. Usually we will examine graphs that correspond to the *minimal* resolution of singularities.

Definition 2.10 *A graph Γ is said to be* **minimal** *if it does not contain internal vertices that have $p_a = 0$ and weight 1.*

Definition 2.11 *For any graph with a nondegenerate quadratic form $F_i \cdot F_k$ (for example elliptic or hyperbolic) we define* **log discrepancies** *a_i as the solutions of a system of linear equations*

$$\sum a_i F_i \cdot F_k = (2p_a(F_k) - 2 - F^2) + (f^{-1}B + \sum F_i)F_k$$

Definition 2.12 *For any graph with a nondegenerate quadratic form $F_i \cdot F_k$ we define* **codiscrepancies** *b_i by the formula $b_i = 1 - a_i$*

Let us explain the meaning of the two previous definitions. The formulae above are equivalent to the following:

$$(K + \sum b_j B_j + \sum b_i F_i)F_k = 0 \quad for \quad any \quad k$$

So if the graph Γ is an elliptic graph, corresponding to some birational morphism $f : Y \to X$ we get the previous definitions 2.3, 2.4. Another situation when we shall use discrepancies and codiscrepancies is the following: X is a surface with numerically trivial $K + B$, $f : Y \to X$ is some resolution. Part of the vertices of Γ correspond to exceptional curves of f and the other part – to strict transforms of certain curves on X.

Definition 2.13 *A graph Γ is said to be* **log canonical (lc) or log terminal (lt)** *with respect to $K + B$ if its log discrepancies $a_i \geq 0$ or $a_i > 0$ respectively.*

The main object into consideration in this paper is a surface X with a divisor $K+B$ that is lc. So will be the corresponding graphs. If we ignore the way B meets the ground graph or assume that all the coefficients of B_j equal 1, then all such graphs are classified in [K] (see also [A3]). They are divided into two classes describing respectively rational and elliptic singularities. In the case of rational singularities all the genera are equal to 0 (and by this reason will be omited), all the edges are simple (of weight 1) the ground graphs are those of types A_n, D_n and E_6, E_7, E_8. If we fix some number N

and consider graphs with weights $\leq N$ then the only infinite series of such graphs are the following

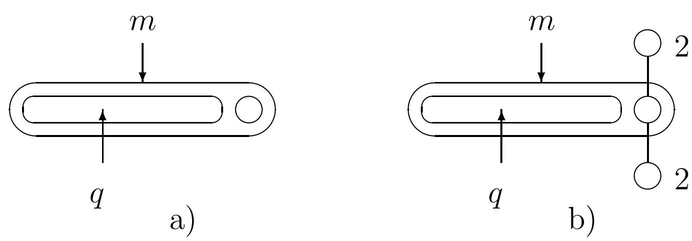


Figure 1

These are typical pictures that we shall use to describe graphs. Long ovals denote chains of vertices. The numbers q and m denote the absolute values of determinants of the submatrices of $F_i \cdot F_k$ that include only rows and columns corresponding to the vertices of the chains. It is very well known ([B], comp.[A3]) that any chain is uniquely determined by a pair of coprime numbers (q, m) with $1 \leq q < m$ and vice versa. In the previous example q and m are any such numbers.

Generally, graphs shall also have external parts that shall be denoted by crossed vertices.

In the case of elliptic singularities one has "circles" of vertices with $p_a = 0$ and a single vertex with $p_a = 1$. B is empty and all the log discrepancies $a_i = 0$, codiscrepancies $b_i = 1$.

Definition 2.14 Du Val graph *is an elliptic graphs with all genera = 0, all weights = 2 and empty external part B. It is well known that the ground graph is then one of the graphs A_n, D_n, E_6, E_7 or E_8.*

Definition 2.15 *We say that a graph Γ' is a subgraph of Γ if all the vertices of Γ' are at the same time vertices of Γ, weights of vertices and edges of Γ' and p_a of vertices in Γ' do not exceed the corresponding weights and p_a in Γ and $F_i' \sum b_j' B_j' \leq F_i' \sum b_j B_j$ for the corresponding vertices.*

The following are easy linear algebra statements.

Lemma 2.16 *Let Γ be a minimal elliptic graph. Then all the log discrepancies $a_i \leq 1$ (codiscrepancies $b_i \geq 0$) and if Γ is not a Du Val graph then $a_i < 1$ ($b_i > 0$).*

Proof: Well known.

Q.E.D.

Lemma 2.17 *Let $\Gamma' \subset \Gamma$, $\Gamma' \neq \Gamma$ be two minimal elliptic graphs and assume that the weights of the vertices are in both graphs the same. Then for the log discrepancies one has $a_i \leq a_i'$ (for codiscrepancies $b_i \geq b_i'$) and if Γ is not a Du Val graph then $a_i < a_i'$ ($b_i > b_i'$). If the weights of Γ' and Γ are different then $a_i \leq a_i'$ assuming that Γ is log canonical.*

Proof: Compare the corresponding systems of linear equations (see [A1], [A3]).

Q.E.D.

Lemma 2.18 *Let $\Gamma' \subset \Gamma$, $\Gamma' \neq \Gamma$ be two graphs such that all the log discrepancies of Γ' $a_i \leq 1$ (codiscrepancies $b_i \geq 0$) and v_0 is a fixed vertex of Γ'. Assume that Γ is hyperbolic and that $\Gamma - v_0$ is elliptic. Then for the log discrepancy of v_0 one has $a_0 \geq a_0'$ ($b_0 \leq b_0'$) assuming that Γ is log canonical.*

Proof: Compare the corresponding systems of linear equations.

Q.E.D.

Corollary 2.19 *Let Γ be a minimal elliptic graph and assume that all the log discrepancies of Γ $a_i \geq c > 0$. Then weights of the vertices are bounded from above by $2/c$.*

Proof: Consider a graph Γ' containing a single vertex of weight n. Then $a' = 2/n$.

Q.E.D.

Corollary 2.20 *Let Γ be a graph as in 2.18 plus let v_0 have weight 1. Assume that the codiscrepancy of v_0 $b_0 \geq c > 0$. Then*

$$\sum_{i \neq 0} F_0 F_i \leq 2 + \frac{2}{c}$$

Proof: Consider a graph Γ' containing a vertex v_0 connected with n vertices of weight 2. Then $b_0' = 2/(n+2)$.

Q.E.D.

2.3 Sequences

Definition 2.21 *Let X be a variety with a log canonical $K+B$.* **A log discrepancy of $\mathbf{K+B}$** *$ld(K+B)$ is a minimal log discrepancy a_i that appears in 2.3 for some birational morphism $f : Y \to X$.*

It is easy to see that $ld(K+B)$ is well defined and is a nonnegative rational number.

Definition 2.22 *Let X be a surface with a log canonical $K+B$.* **A partial log discrepancy of $\mathbf{K+B}$** *$pld(K+B)$ is a minimal log discrepancy a_i that appears in 2.3 for the special birational morphism $h : \widetilde{X} \to X$, where $\widetilde{X}$ is the minimal resolution of singularities.*

Definition 2.23 *Let $\xi = \{X^{(n)}, K+B^{(n)} | n = 1, 2...\}$ be a sequence of surfaces. Then we define* $\mathbf{ld}(\xi)$ *and* $\mathbf{pld}(\xi)$ *as the* **sequences** *of real numbers $\{ld(K+B^{(n)})\}$ and $\{pld(K+B^{(n)})\}$ respectively.*

Definition 2.24 *Let $\xi = \{X^{(n)}, K+B^{(n)} | n = 1, 2...\}$ be a sequence of surfaces. Then we define* $\mathbf{LD}(\xi)$ *and* $\mathbf{PLD}(\xi)$ *as the* **subsets** *of real numbers $\{ld(K+B^{(n)})\}$ and $\{pld(K+B^{(n)})\}$ respectively.*

Definition 2.25 *We define ld, pld, LD, PLD for graphs in the same way as we have done it for surfaces.*

Definition 2.26 *Let $B = (b_1, b_2...b_s)$ an $B' = (b'_1, b'_2...b'_t)$ be two groups of numbers. One says that $B \le B'$ if*

(i) $s \ge t$

(ii) for every $j = 1...t$ $b_j \le b'_j$

If, in addition, one of the inequalities in (i) or in (ii) for some index j_0 is strict, one says that $B < B'$.

Remark 2.27 *Because of the part (i) of 2.26 when considering a nondecreasing sequence $B^{(n)}$ we can always assume, passing to a subsequence, that the lengths of $B^{(n)}$ are in fact the same.*

2.4 Log Del Pezzo surfaces

Definition 2.28 *A normal surface X is said to be a Del Pezzo surface if $-K$ is an ample $\mathbf{Q}$-divisor.*

The following is an simple lemma, see [A-N],[N] for the proof which is especially easy if K is lt or lc.

Lemma 2.29 *Let X be a log Del Pezzo surface an $h : \widetilde{X} \to X$ be the minimal resolution of singularities. Then*

(i) the Kleiman-Mori cone of effective curves $NE(\widetilde{X})$ is generated by finitely many extremal rays

(ii) if $X \neq \mathbf{P}^2, \mathbf{F}_n$ (minimal rational surface) then all the extremal rays are generated by exceptional curves of f and (-1)-curves.

Lemma 2.30 *Let X be a Del Pezzo surface and assume that K is lc. Then X is one of the following:*

(i) a rational surface with rational singularities

(ii) a generalized cone over a smooth elliptic curve

Proof: Let $h : \widetilde{X} \to X$ be a minimal desingularization. $\widetilde{X}$ is a smooth surface and clearly $h^0(NK_{\widetilde{X}} = 0$ for any $N > 0$, so $\widetilde{X}$ is ruled.

Assume that X has a nonrational singularity. Then by the classification of log canonical singularities $\widetilde{X}$ contains an elliptic curve or a circle of rational curves F_0 that is disjoint from other curves, exceptional for h. If $\widetilde{X}$ is a locally trivial $\mathbf{P}^1$-bundle then F_0 should be an exceptional section of this bundle and should be smooth. In this case X is a generalized cone. Otherwise F_0 should intersect a curve E with $E^2 < 0$ that lies in the fiber of a generically $\mathbf{P}^1$-bundle giving the structure of a ruled surface and such that E is not exceptional for h. By 2.29 E is a (-1)-curve. The latter is impossible since $-h^*K = -K_{\widetilde{X}} - F_0 - ...$ and therefore $-h^*K \cdot E \leq 0$.

Now let us assume that X has only rational log canonical singularities. By the classification again one has $-h^*K = -K_{\widetilde{X}} - F_0 - \sum b_i F_i$, $0 \leq b_i < 1$

and F_0 is a disjoint union of smooth rational curves. Since $-h^*K$ is big, nef, the Kawamata-Fiehweg vanishing gives

$$h^1(\widetilde{X}, -F_0) = h^1(\widetilde{X}, K_{\widetilde{X}} + \sum b_i F_i + (-h^*K)) = 0$$

and from the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathcal{X}}}(-\mathcal{F}_0) \to \mathcal{O}_{\widetilde{\mathcal{X}}} \to \mathcal{O}_{\mathcal{F}_0} \to 0$$

one gets $h^1(\mathcal{O}_{\widetilde{\mathcal{X}}}) = 0$. Therefore $\widetilde{X}$ and X are rational surfaces.

Q.E.D.

3 Local case: elliptic log canonical graphs

In this section we consider only local situation. X is a neighbourhood of a surface point P and all the components of B pass through P.

Theorem 3.1 (Local boundness) *Let $X, K+B$ be as above a neighbourhood of a surface point P with lc $K+B$ and all of B_j pass through P. Then $\sum b_j \le 2$.*

Proof: Proved in [A-K] for the n-dimensional case with a bound n.

3.1 Minimal resolution

Theorem 3.2 (Local partial ascending chain condition) *Let $\xi = \{X^{(n)}, K+B^{(n)}\}$ be a sequence of surfaces such that*

(i) $K + B^{(n)}$ is lc

(ii) $B^{(n)}$ is a nondecreasing sequence (for example, constant)

Then every increasing subsequence in pld(ξ) terminates.
If, in the addition, one has

(iii) $B^{(n)}$ is an increasing sequence

then every nondecreasing subsequence in pld(ξ) terminates.

We prove 3.2 in several steps.

Step 1 *One can assume that $K + B^{(n)}$ and moreover K are lt.*

Proof: Indeed, if in (ii) $K + B^{(n)}$ is not lt, then $pld(K + B^{(n)}) = 0$ but we are looking for increasing subsequences of $pld(\xi)$. In (iii) if $K + B^{(n)}$ is not lt, then there exists a partial resolution $f : Y \to X$ with a single exceptional divisor F such that the corresponding log discrepancy $a = 0$. Then

$$K_Y + F + f^{-1}B = f^*(K + B)$$

and the log adjunction formula for F (see [SH3]) yields

$$\sum \frac{k - 1 + \sum l_j b_j}{k} = 2$$

for some positive integers k, l_j. It is easy to see that if $B^{(n)}$ are increasing then the sequence should terminate.

Q.E.D.

Step 2 *By the previous step we can assume that there is a constant ε so that for every n $pld(K + B^{(n)}) > \varepsilon$. Then we prove the following*

Lemma 3.3 *All the lt elliptic graphs with $pld(K + B^{(n)}) > \varepsilon$ and $b_j > \varepsilon$ can be described as follows:*

(i) finitely many graphs (that includes the way B_j intersect F_i)

(ii) the graphs given on the next picture, where there are only finitely many possibilities for the chains of vertices, denoted by ovals and for the ways B_j meet that vertices

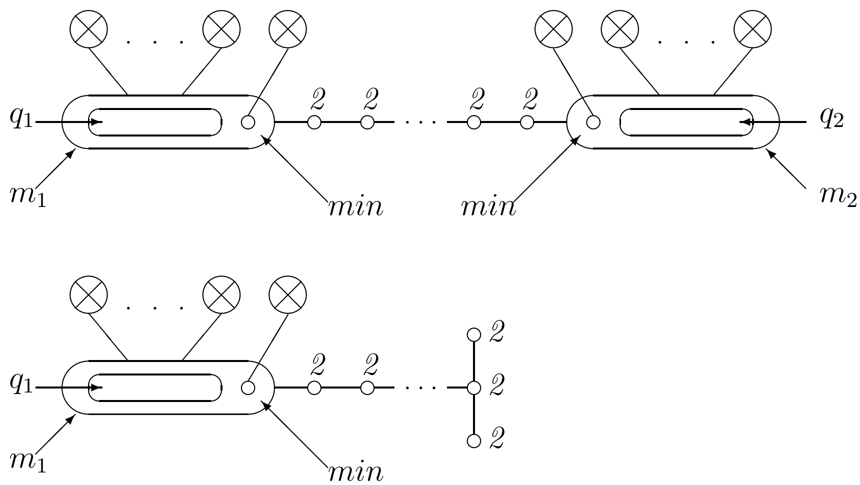


Figure 2

Moreover, the log discrepancies of any of suchs graphs satisfy the following inequality

$$pld(K+B) \geq \frac{1-\sum l_j b_j}{m-q}$$

where $\bar{b}_j = \lim b_j$ *and tend to this number as the chain of 2's gets longer and longer. Here* $l_j = \sum (B_j \cdot F_i) r_i$*, where* r_i *is the determinant of the short subchain of the ground graph, "cut off" by the vertex* v_i*.*

Proof: By 2.19 the weights of vertices in the graph Γ are bounded by $2/\varepsilon$. Therefore, sacrificing finitely many graphs we can assume that Γ is one of the graphs on Fig.1.

First, assume that we are in the case a) of Fig.1, i.e. Γ is a chain. Consider the sequence of log discrepancies of vertices in this chain. By [A3]

$$a_{i-1} - 2a_i + a_{i+1} = (w_i - 2)a_i + \sum b_j B_j F_j \geq (w_i - 2 + \sum B_j F_i)\varepsilon,$$

therefore the graph of this function is concave up and, unless $w_i = 2$ and all $B_j F_i = 0$, it is not a straight line but is "very concave up". Now by 2.16 the

discrepancies $a_i \leq 1$. This implies that all the chains are those on Fig.2 with only finitely many possibilities for the ovals and with the chains of 2's of an arbitrary length A. Also, omiting finitely many graphs, we can assume that the minimal log discrepancy is achieved at one of the two vertices, where the arrows point out.

Now we use an explicit formula for the log discrepancies of those vertices which follows easily from 3.1.8, 3.1.10 of [A3].

Define $\alpha_1 = 1 - \sum l_j^{(1)} b_j$ for the left part of the chain, the meaning of l_j being explained in the formulation of the statement, $\alpha_2 = 1 - \sum l_j^{(2)} b_j$ be the corresponding expression for the right par, and let A be the length of the chain of 2's. Then

$$a_1 = \frac{\alpha_1(A(m_2 - q_2) + m_2) + \alpha_2 q_1}{A(m_2 - q_2)(m_1 - q_1) + m_2(m_1 - q_1) + q_1(m_2 - q_2)}$$

or

$$a_1 = \frac{\frac{\alpha_1}{m_1 - q_1}(A + \frac{m_2}{m_2 - q_2}) + \frac{\alpha_2}{m_2 - q_2}\frac{q_1}{m_1 - q_1}}{A + \frac{m_2}{m_2 - q_2} + \frac{q_1}{m_1 - q_1}}$$

with the symmetric expression for a_2.

One can note that

1. if $\frac{\alpha_1}{m_1 - q_1} \leq \frac{\alpha_2}{m_2 - q_2}$ then $\frac{\alpha_1}{m_1 - q_1} \leq a_1 \leq \frac{\alpha_2}{m_2 - q_2}$
2. $\lim_{A \to \infty} a_1 = \frac{\alpha_1}{m_1 - q_1}$

and these two observations complete the proof in the case a) of Fig.1. The case b) of Fig.1 is handled similarly. Let us mention only that in the latter case there is only one possible vertex for the minimal log discrepancy which is given by the formula

$$a_1 = \frac{\alpha_1}{m_1 - q_1},$$

so this case can be treated formally as a subcase of a) with $\alpha_2 = 0$ and $m_2 = q_2$.

Q.E.D.

Step 3 *The lemma 3.3 implies 3.2.*

Proof: For any fixed graph Γ if the coefficients of the external part B increase, then by 2.17 log discrepancy $pld(K+B)$ decreases. Therefore, we can consider only case (ii) of 3.3. Passing to a subsequence we can assume that all the graphs are of the same type and the length of the sequence of 2's increases. But then

$$pld(K+B^{(n)}) \geq \frac{1-\sum l_j \bar{b}_j}{m-q}$$

and

$$\lim pld(K+B^{(n)}) = \frac{1-\sum l_j \bar{b}_j}{m-q}$$

where $\bar{b}_j = \lim b_j$, and we are done.

Q.E.D.

Corollary 3.4 *If $B = \emptyset$, then 3.3 says that the set of minimal log discrepancies satisfies the ascending chain condition and the only limit points are 0 and $1/k$, $k = 2, 3...$*

Remark 3.5 *The statement 3.4 is due to V.V.Shokurov (unpublished).*

3.2 General case

Later we shall use the local ascending chain condition in the just proved form, i.e. for the minimal resolution of singularities. But the minimal resolution of singularities of X is not necessarily a resolution of singularities for $K+B$, because $supp(f^{-1}B \cup F_i)$ can have nonnormal intersections. Below we prove the statement, corresponding to 3.2 but for $ld(\xi)$ instead of $pld(\xi)$. We first consider the case when $X^{(n)}$ are nonsingular and then combine our arguments to treat the general situation.

Theorem 3.6 *Let $\xi = \{X^{(n)}, K+B^{(n)}\}$ be a sequence of* **nonsingular** *surfaces such that*

(i) $K+B^{(n)}$ is lc

(ii) $B^{(n)}$ is a nondecreasing sequence (for example, constant)

Then every increasing subsequence in $ld(\xi)$ terminates.
If, in the addition, one has

(iii) $B^{(n)}$ is an increasing sequence

then every nondecreasing subsequence in $ld(\xi)$ terminates.

Proof: As above, we can assume that that $K + B^{(n)}$ are in fact lt.

Now let us find out what happens with a nonsingular surface X with $K+B$ after a single blow up $f : X \to Y$ at the point P, F as usually denotes the exceptional divisor of f. The answer is evident:

$$f^*(K + \sum b_j B_j) = K_Y + \sum b_j f^{-1} B_j + (-1 + \sum mult_P B_j b_j) F \tag{1}$$

and the condition $a > 0$ translates to $-1 + \sum mult_P B_j b_j < 1$. If $-1 + \sum mult_P B_j b_j \leq 0$, then for any further blowups all the log discrepancies $a_i \geq a$, so they are irrelevant in finding the minimal log discrepancy and $K+B$ is lt. However, if this is a positive number, some negative log discrepancies can appear on the following steps.

Now let $f : X \to Y$ be a composite of several blow ups. One gets

$$f^*(K + \sum b_j B_j) = K_Y + \sum b_j f^{-1} B_j + \sum (-s_i + \sum t_{ik} b_k) F_i \tag{2}$$

with some nonnegative integers s_i, t_{ik} and $s_i \leq \rho(Y/X)$. The corresponding log discrepancies are given by

$$a_i = 1 + s_i - \sum t_{ik} b_k$$

and for fixed s_i and nondecreasing/increasing b_j they evidently form a non-increasing/decreasing sequences. Note that there are only finitely many such sequences with $a_i \geq 0$. Therefore 3.2 follows from the following lemma.

Lemma 3.7 *With the assumptions as above, there is a constant $N(\xi)$ so that for every surface $X^{(n)}$ in ξ there exists a birational morphism $g : Y^{(n)} \to X^{(n)}$ such that*

(i) $\rho(Y^{(n)}/X^{(n)}) < N(\xi)$

(ii) the minimal log discrepancy $ld(K + B^{(n)})$ is one of the log discrepancies of g.

Proof: Let us remind that we are in the local situation, so $X^{(n)}$ is a neighbourhood of a (nonsingular) point P. Let $f : Z^{(n)} \to X^{(n)}$ be a single blow up at P. If in the formula 1 the number $C = -1 + \sum mult_P B_j b_j$ is positive and on $Y^{(n)}$ the strict transforms of B_j intersect at one point and have the same multiplicities as on $X^{(n)}$, then by the formula 1 on the second blowup codiscrepancy of the exceptional divisor equals $2C$, after the third blowup $3C$ and so on (and it should be ≤ 1). Since $B^{(n)}$ is nondecreasing, there exists a constant $\varepsilon(\xi)$ so that for any $-1 + \sum m_j b_j > 0$, one also has $-1 + \sum m_j b_j > \varepsilon(\xi)$. The conclusion is that there exists a number N_1, depending on ξ, so that after $N_1(\xi)$ blowups the configuration of B_j simplifies in some way: either the number of curves, passing through the points, or the multiplicities at those points get smaller; or all the further blowups are irrelevant in finding the minimal discrepancy.

Let $X^{(n)\prime}$ be $X^{(n)}$ with blown up points, $K + B^{(n)\prime} = f^*(K + B^{(n)})$. Note that the coefficients of $B^{(n)\prime}$ are still nonnegative numbers. At the neighbourhood of any point of $X^{(n)\prime}$ $B^{(n)\prime}$ consists of several curves $B_j + \leq 2$ nonsingular curves F_i with coefficients, given by the formula 2 and hence, nondecreasing, and from the finite list of possible combinations. Now we can find the next number $N_2(\xi)$ so that after $N_2(\xi)$ blowups the configuration of ($B_j + \leq 2$ nonsingular curves) simplifies even further. By induction we get the desired result.

Q.E.D.

And finally we prove

Theorem 3.8 (Local ascending chain condition) *Let $\xi = \{X^{(n)}, K + B^{(n)}\}$ be a sequence of surfaces such that*

(i) $K + B^{(n)}$ is lc

(ii) $B^{(n)}$ is a nondecreasing sequence (for example, constant)

Then every increasing subsequence in $ld(\xi)$ terminates.
If, in the addition, one has

(iii) $B^{(n)}$ is an increasing sequence

then every nondecreasing subsequence in $ld(\xi)$ terminates.

Proof: By 3.3 all the singularities with $ld(K+B) \geq \varepsilon$ are divided into finite number of series + finite number of graphs. The latters are taken care by 3.6.

So all we have to do is to consider one of the graphs on Fig.2 with the chain of 2's that is getting longer and longer. And a simple calculation shows that for all except finitely many graphs the minimal log discrepancy is in fact one of log discrepancies of $h : \widetilde{X} \to X$.

Q.E.D.

4 Special hyperbolic log canonical graphs

Set-up In this section Γ or $(X, K+B)$ always denote the following:

Definition 4.1 *We say that a graph* Γ *is* **special hyperbolic** *if*

(i) Γ *is hyperbolic and connected*

(ii) all the vertices have $p_a = 0$*, there is a special vertex* v_0 *of weight 1, all other vertices* v_i *have weights* ≥ 2

(iii) $\Gamma - v_0$ *is elliptic*

(iv) as usually, Γ *may have an external part* $B = \sum b_j B_j$

Such graphs naturally appear when one considers a minimal resolution of singularities of a Del Pezzo surface X with $\rho(X) = 1$ and B_0 being a (-1)-curve on the resolution.

Theorem 4.2 (Local-to-global ascending chain condition) *Let* $\xi = \{X^{(n)}, K + B^{(n)}\}$ *be a sequence of special hyperbolic graphs with a chosen vertex* B_0 *such that*

(i) $K + B^{(n)}$ *is lc*

(ii) $\{B^{(n)}\}$ *is an increasing sequence, moreover,* $\{b_0^{(n)}\}$ *is an increasing sequence*

(iii) all the log discrepancies $a_i \geq 1 - \bar{b}_0 = 1 - \lim b_0$

(iv) $K + B^{(n)}$ *is numerically trivial*

Then ξ terminates.

Proof:
Case 1: $\bar{b}_0 = \lim b_0 = 1$.

From [A3] it follows that if b_0 is close enough to 1, then all the singularities (that is, the connected components of $\Gamma - v_0$) and the ways the components of B meet F_i are exhausted by the following list:

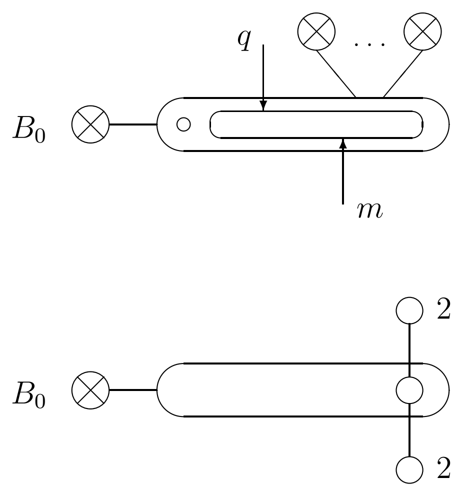


Figure 3

The next step is a formula for the coefficient b_0 that follows from the explicit calculations of [A3]:

$$b_0 = \frac{\sum_{s=1}^{N} \frac{q_s+\alpha_s}{m_s} - (N-1)}{\sum_{s=1}^{N} \frac{q_s}{m_s} - 1} = 1 - \frac{(N-2) - \sum_{s=1}^{N} \frac{\alpha_s}{m_s}}{\sum_{s=1}^{N} \frac{q_s}{m_s} - 1} \tag{3}$$

with denominator > 0, where N is a number of connected components of $\Gamma - v_0$. Here $\alpha_s = 1 - \sum l_j^s b_j$ as in 3.3. We consider the second case of the figure 3 formally as a subcase of the first one with $q = m$ and $\alpha = 0$.

Note that by 2.20 a number of graphs that $v_0^{(n)}$ is connected with in the sequence is bounded.

For any fixed N the conditions $\lim b_0 = 1$ and $b_0 < 1$ imply

$$\sum_{s=1}^{N} \frac{\alpha_s}{m_s} < N-2 \quad and \quad \lim \sum_{s=1}^{N} \frac{\alpha_s}{m_s} = N-2.$$

We can assume that some of m_s are fixed and others tend to infinity. For the latters $\frac{\alpha_s}{m_s} \to 0$ and $\frac{\alpha_s}{m_s} > 0$. This is so by 3.3 (here it is important again that there is a constant $\varepsilon(\xi)$ so that $\sum m_j b_j - 1 > 0$ implies $\sum m_j b_j - 1 > \varepsilon(\xi)$) and by 3.1. So we can assume that $\sum_{s=1}^{M} \frac{\alpha_s}{m_s} < N-2$ and

$$\lim \sum_{s=1}^{M} \frac{\alpha_s}{m_s} = \lim \sum_{s=1}^{M} \frac{1 - \sum l_j^s b_j}{m_s} = N-2$$

with $m_1 ... m_M$ being fixed. But this definitely gives a contradiction. Note that $\sum l_j^s b_j \le 2$ by 3.1, so there are only finitely many possibilities for l_j^s.

Finally, for $N \ge 5$

$$b_0 \le 1 - \frac{(N-2) - \sum \frac{1}{m_s}}{N-1} \le 1 - \frac{\frac{N}{2} - 2}{N-1} \le \frac{7}{8}$$

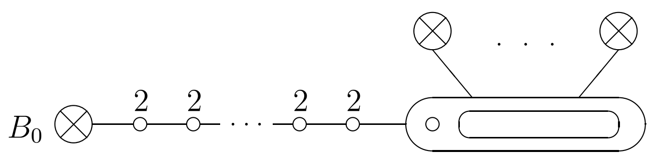
and we are done.

Case 2: $\bar{b}_0 = \lim b_0 < 1$.

Since all the log discrepancies $a_i \ge \varepsilon = 1 - \bar{b}_0$, the only infinite series of connected components of $\Gamma - v_0$ are given by 3.3. Moreover, for the minimal log discrepancies there

$$\lim \min a_i \le \frac{1 - \sum (B_0 F_i) r_i b_0}{m-k}$$

and this should be not less than $1 - \bar{b}_0$. As a conclusion, all the infinite series are given by



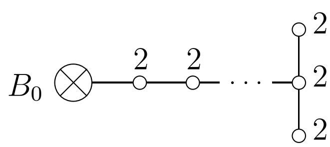


Figure 4

Now we would like to use a variant of the formula 3. However, B_0 can intersect finitely many types of graphs arbitrarily. Still, for any fixed combination, if b_j increase, b_0 decreases. The situation is exactly the opposite to the one of elliptic graphs since the signature of the quadratic form is now $(1, n-1)$ instead of $(0, n)$ and the graph $\Gamma - v_0$ is still elliptic (cf. 2.20).

All the said above implies that for b_0 there are only finitely many possible expressions of the form

$$b_0 = 1 - \frac{C_1 + \sum C_2^j b_j - \sum_{s=1}^N \frac{\alpha_s}{m_s}}{C_3 + \sum_{s=1}^N \frac{q_s}{m_s}}$$

with fixed $C_1, C_2^j, C_3, m_s - q_s$, $m_s \to +\infty$, $C_2^j \geq 0$ and the denominator > 0. Simplifying,

$$1 - b_0 = \frac{C_1 + \sum C_2^j b_j - \sum_{s=1}^N \frac{\alpha_s}{m_s}}{C_3' - \sum_{s=1}^N \frac{m_s - q_s}{m_s}}$$

Now $\lim b_0 = \bar{b}_0$ implies $(C_1 + \sum C_2^j \bar{b}_j)/C_3' = 1 - \bar{b}_0$. And, finally, the inequalities $\frac{\alpha_s}{m_s - q_s} \geq 1 - \bar{b}_0$ and $C_2^j \geq 0$ imply that$1 - b_0 \leq 1 - \bar{b}_0$, i.e. $b_0 \geq \bar{b}_0$ that gives a contradiction.

Q.E.D.

Remark 4.3 *As the proof shows, 4.2 is not true without the assumption (iii).*

5 Global case

Theorem 5.1 (Global boundness) *Let X be a surface with a lc divisor $K+B$ and assume that $f : X \to Y$ is a contraction of an extremal ray such that $K+B$ if f-nonpositive. Let $B^+ = \sum b_j^+ B_j^+$ contain all the components in B that are f-positive. Then*

(i) if $\dim Y = 2$, $\sum b_j^+ \leq 2$

(ii) if $\dim Y = 1$, $\sum b_j^+ \leq 2$

(iii) if $\dim Y = 0$, $\sum b_j^+ \leq 3$

Proof: (i) follows from 3.1, because $K_Y + f(B)$ is also lc. (ii) is clear: if B^+ is not empty, then $-K$ should be negative on the general fiber, so a general fiber F is isomorphic to $\mathbf{P}^1$ and $\sum b_j^+ \leq B^+F \leq -KF = 2$.

In the case (iii) if X is nonsingular, then $X \simeq \mathbf{P}^2$ and the statement is evident. If X is singular, consider a partial resolution $g : Z \to X$, dominated by the minimal resolution and such that $\rho(Z) = \rho(X) + 1 = 2$. Then by 2.29 there is a second extremal ray and $g^*(K+B) = K_Z + B_Z$ is nonpositive with respect to this extremal ray. Since every curve on Z is positive with respect to at least one of the extremal rays, (iii) with the bound 4 follows immediately. To get the bound 3 is an easy exercise.

Q.E.D.

Remark 5.2 *5.1(iii) is also proved in [A-K] for arbitrary dimension with a bound $n+1$.*

Theorem 5.3 (Global ascending chain condition) *Let $\xi = \{X^{(n)}, K + B^{(n)}\}$ be a sequence of surfaces such that*

(i) $K + B^{(n)}$ is lc

(ii) $B^{(n)}$ is an increasing sequence

(iii) $K+B^{(n)}$ is numerically trivial

Then ξ terminates.

Proof of 5.3 will be given in several steps.

Step 1 *One can assume that all the surfaces $X^{(n)}$ are Del Pezzo surfaces with $\rho(X^{(n)})=1$.*

Proof: We can assume that the lengths of the groups $B^{(n)}$ in the sequence ξ are constant and that b_1 always increases. Now consider a divisor $K+B-\varepsilon B_1$ on $X^{(n)}$. Note here that B_1 is **Q**-factorial by the classification of log canonical singgularities. It is lc and is not numerically effective and if $B_1^2 \leq 0$ then $(K+B-\varepsilon B_1)B_1 \geq 0$. Therefore either $\rho(X^{(n)})=1$ and then $X^{(n)}$ is a Del Pezzo surface with lc $K+B$ or there is an extremal ray that does not contract B_1. If the contraction is birational, we make it and repeat the same procedure again. If it is a fibration, the claim follows from the corresponding 1-dimensional statement.

Q.E.D.

Remark 5.4 *The argument works in the 3-dimensional case as well.*

Step 2 *One can assume that there are only finitely many different types of graphs of singularities that the increasing components of $B^{(n)}$ are passing through.*

Proof: As usually, we can assume that the groups $B^{(n)}$ have the same length. Now consider the set $PLD(\xi)$. By 3.2 this set satisfies the ascending chain condition and has at least one limit point. Let l be the minimal limit point of $PLD(\xi)$. Fix the number C so that all $b_j \geq C$. If the surfaces in ξ contain singularities that correspond to infinitely many elliptic graphs, then by 3.3 $l \leq 1-C$. Passing to a subsequence we can assume that a sequence of minimal log discrepancies, which we shall denote $\{a_s^{(n)}\}$ is a decreasing sequence and $\lim a^{(n)_s} = l$ (the sequence of codiscrepancies is increasing and $\lim b_0^{(n)_s} = 1-l \geq C$.

Now consider a partial resolution $f: Y^{(n)} \to X^{(n)}$ which is dominated by the minimal desingularization and which blows up exactly the curve $B^{(n)_s}$. Then

$$f^*(K+B^{(n)}) = K_Y + f^{-1}B^{(n)} + b^{(n)_s}B^{(n)_s}$$

The surface $Y^{(n)}$ has Picard number 2 and by 2.29 there is a second extremal ray, corresponding to a (-1)-curve on $\widetilde{Y}=\widetilde{X}$. Let $g: Y^{(n)} \to X'^{(n)}$ be the contraction of this second extremal ray. If g is a fibration then restricting of $K_Y + f^{-1}B^{(n)} + b^{(n)_s}B^{(n)_s}$ on the general fibre of g readily gives a contradiction. Hence, we shall assume that g is a birational morphism. A divisor $K+B'^{(n)} = g_* f^*(k+B^{(n)})$ is lc and numerically trivial, $B'^{(n)}$ has either the same number of components as $B^{(n)}$ or one more, and, after passing to a subsequence, $B'^{(n)}$ is an increasing sequence.

A morphism g can contract one of the components of $B^{(n)}$ and we can assume that it is always, say, B_0. However, by 4.2 and 3.2 the sequence $\{b_0^{(n)}\}$ cannot be an increasing sequence with $\lim b_0^{(n)} \geq 1-l$. Therefore, changing the sequence $\xi = \{X^{(n)}\}$ by a new sequence $\xi' = \{X'^{(n)}\}$, we are gaining a new component with increasing coefficient that has the limit $1-l \geq C$. Note that for a new minimal limit point l' of $PLD(\xi')$ one has $l' \geq l$. This is so because a minimal desingularization of $X'^{(n)}$ is dominated by the minimal desingularization of $X^{(n)}$ and $K+B^{(n)}$, $K+B'^{(n)}$ both are numerically trivial, so $PLD(\xi')$ is a subset in $PLD(\xi)$.

Repeating the procedure, we get one more component and so on. After k steps the sum of the coefficients in $B^{(n)}$ will be greater than kC. This eventually will get into the contradiction with 5.1.

Q.E.D.

Step 3 *One can assume that all the surfaces $X^{(n)}$ are isomorphic to each other.*

Proof: By 2.30 a surface $\widetilde{X^{(n)}}$ is either a locally trivial $\mathbf{P}^1$-bundle with a section which is a smooth elliptic curve or a rational surface with rational singularities. In the former case the statement follows from the 1-dimensional analog by restricting B to the fiber of the fibration. Now assume we are in the latter case. By the previous step, there exists a constant $N(\xi)$ so that for the increasing component B_1 of B NB_1 is Cartier. Hence for any curve D on $X^{(n)}$

$$-KD = \sum b_j B_j D \geq b_1/N \geq C/N$$

Now theorem 2. 3 of [A1] states that for all such surfaces $\rho(\widetilde{X})$ is bounded. Therefore one can get $X^{(n)}$ by blowing up finitely many points from the minimal rational surface $\mathbf{F}_k$. Threfore there are only finitely many possibilities for the graph of exceptional curves on $X^{(n)}$ except for the fact that

one weight k can be arbitrary. Now if B_1 does not lie in the fiber for infinitely many n we prove the statement restricting a numerically trivial divisor $K + \sum b_j B_j + \sum b_i F_i$ to the fiber and using 3.2. Otherwise (recall that $\rho(X) = 1$) B_1 on X should pass through the singularity which graph contains the exceptional curve of $\mathbf{F}_k$. By the previous step k is bounded. Hence we can assume that the surfaces $X^{(n)}$ belong to a bounded family and it is enough to consider only finitely many of them.

Q.E.D.

Remark 5.5 *Theorem 2.3 in [A1] is stated for log terminal singularities. But in fact the proof is exactly the same for rational log canonical singularities.*

Step 4 *5.3 follows.*

Proof: Indeed, there are only finitely many possibilities for effective Weil divisors B_j.

Q.E.D.

The following example shows that 5.3 is not true without the assumption (i).

Example 5.6 *Consider a sequence of surfaces $X^{(n)}$ so that $\widetilde{X}^{(n)} = \mathbf{F}_n$ and $B^{(n)} = (1 - 1/n)F_1 + 3/4(F_2 + F_3 + F_4)$ where F_1 is an image of the infinite section of $\mathbf{F}_n$, $F_{2,3,4}$ are fibres. Note that $K + B^{(n)}$ is numerically trivial but is not lc.*

References

[A-K] A.Agrassi, J.Kollár. *Reduction to the special log flips.* In: J.Kollár et al. Flips and abundance for algebraic threefolds. To appear in Asterisque.

[A-N] V.Alexeev, V.V.Nikulin. *Classification of Del Pezzo surfaces with log terminal singularities of index ≤ 2 and involutions on K3 surfaces.* Soviet Math. Dokl. 39 1989.

[A1] V.Alexeev. *Fractional indices of log Del Pezzo surfaces.* Math USSR Izvestia, 33 1989, 613-629.

[A2] V.Alexeev. *Theorems about good divisors on log Fano variety of index* $r > n - 2$. Lecture Notes in Math, 1479. 1-9.

[A3] V.Alexeev. *Classification of log canonical surface singularities: arithmetical proof.* In: J.Kollár et al. Flips and abundance for algebraic threefolds. To appear in Asterisque.

[B] E.Briescorn. *Rationale Singularitäten komplexer Flächen.* Invent. Math. 4 1968, 336-358.

[M] S.Mori. *Threefolds whose canonical bundle is not numerically effective.* Ann of Math 116 1982, 133-176.

[K] Y.Kawamata. *The crepant blowing-up of 3-dimensional canonical singularities and its application to the degeneration of surfaces.* Ann of Math 127 1988, 93-163.

[KMM] Y.Kawamata, K.Matsuda, K.Matsuki. *Introduction to the minimal model problem.* in: Algebraic Geometry Sendai 1985, Adv. Studies Pure Math. (1987), 263-360.

[N] V.V.Nikulin. *Log Del Pezzo surfaces III.* Math. USSR Izvestia 35 1990, 657.

[SH1] V.V.Shokurov. *On the closed cone of curves of algebraic 3-folds.* Math. USSR Izvestia 24 1985, 190-198.

[SH2] V.V.Shokurov. In: Birational geometry of algebraic varieties: open problems. 23 Taneguchi intnl symposium, 1988, Katata.

[SH3] V.V.Shokurov. *3-fold log flips.* To appear in Math. xUSSR Izvestia.